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PHILOSOPHICAL AND AXIOMATIC GROUNDING OF FUZZY THEORY

İ. Burhan TÜRKŞEN

TOBB ETÜ, Mühendislik Fakültesi, Endüstri Mühendisliği Bölümü, Söğütözü 06560, Ankara
Department of Mechanical and Industrial Engineering, University of Toronto, Toronto, Ontario, M5S 3G4, Canada
bturksen@etu.edu.tr

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ABSTRACT

At times, it is essential to re-affirm the philosophical and axiomatic foundations of Fuzzy Theory in order to search and discover new avenues of research and to shed a new light onto basic assumptions. For such a purpose, first, the perspectives of Pierce and Zadeh are reviewed with regards to determinacy and indeterminacy. Secondly, the ontological and epistemological foundations of both the Classical and Fuzzy theories are briefly noted from the perspective of a theoretical inquiry. Thirdly, axiomatic positions are re-stated for: 1) classical set and logic theories, 2) fuzzy set and two-valued logic theories,i.e., Type I fuzzy theory, and then 3) a fuzzy interpretation of Meta-Linguistic Axioms are investigated to reveal part of the foundational underpinnings of Interval-Valued Type II fuzzy theory.

Keywords: Fuzzy theory, it's philosophy, it's axioms, ontological, epistemological, type I and II structures

BULANIK TEORİNİN BELİTLERE DAYALI FELSEFİK TEMELLERİ

ÖZET

Zaman zaman, yeni araştırma yönleri aramak, keşfetmek ve temel varsayımlara yeni bir ışık tutmak için, Bulanık Teorinin felsefi ve belitlere dayanan temellerini tekrarlamak önemlidir. Bu gaye nedeniyle, önce, Pierce ve Zadeh'nin görüş açılarının belirli ve belirsizlik yönlerini yeniden incelemek gerekir. İkinci olarak teorik sorgulama perspektifinden, Klasik ve Bulanık teorilerin yaratılış ve bilgi kuramı temellerine kısaca işaret ediyoruz. Üçüncü olarak, Aralık-Değerli Tip II bulanık teorinin temel desteklerini ortaya çıkaran 1) klasik küme ve mantık teorileri, 2)bulanık küme ve iki-değerli mantık teorileri, Tip I bulanık teori ve 3) Dil-Bilim Ötesi Belitlerin bulanık yorumunu tetkik ediyoruz.

Anahtar Kelimeler: Bulanık teori, felsefesi, belitleri, yaratılış, bilgi kuramı, tip I ve II yapıları

INTRODUCTION

In this philosophical and axiomatic grounding of "Fuzzy Theory", it is important to understand an in-depth association of the essential concepts that were treated by Charles S. Peirce and Lotfi A. Zadeh. Their perspectives ought to be expose with their essential concerns underpinning the "indeterminacy and determinacy" of "symbols" from Peirce and "meaning representation" of "words" in Computing With Words, CWW, from Zadeh. Next, the ontological and epistemological foundations of both the Classical and Fuzzy theories need to be briefly noted from the perspective of a theoretical inquiry. As well, a treatment of both the Classical and Fuzzy Theories need to be made from an axiomatic grounding. Finally one ought to reveal an assessment of the "Meta-Linguistic Axioms" from the perspective of CWW.

Peirce and Zadeh

Charles S. Peirce's (1867) logic of mathematics characterize all varieties of indeterminacy and determinacy that affect either the breath (reference, denotation, extension) or depth (sense, connotation, intension) of symbols. Indefiniteness and definiteness (in breath) and precision (in depth) are part of the logic of vagueness (Brock, 1979; Hoopes, 1991). Hence the inter-connection to Zadeh`s (1996-2001) CWW exposing "precisiated meaning representation of words and symbols that represent them. Peirce's discussion pre-supposes that every symbol is capable of determining an interpretant symbol and that symbols are at least potentially general.

Thus in general a symbol S is indeterminate
iff $(\exists P)\sim(S \text{ is } P \text{ OR } S \text{ is } \sim P)$.

Hence, Locke's famous idea of the triangle is stated as:

"It is not the case that a triangle in general is scalene or that is it not scalene."

Peirce indicates, such examples must be understood in terms of universal and affirmative predications. Hence Peirce suggests that we substitute universal "subjects," i.e., "every triangle" or "any triangle," for S.

As a result, the sentence defining indeterminacy is satisfied:

For it is false that "any (every) triangle is scalene and also false that any (every) triangle is not scalene."

In Zadeh`s sense of CWW, the symbol S is determinate to a degree in at least one respect
iff $(\exists P)$ $(S \text{ is } P \text{ OR } S \text{ is } \sim P)$.

Thus for example the term "man" is determinate with respect to the term "animal" because it is true "Any (every) man is an animal to a degree" and false that "No man is an animal to a degree" (every man is non animal to a degree).

On the other hand, consider, for example, the fact that "man" turns out to be indeterminate with respect to "sagacity." But "man has a degree of sagacity" in CWW approach. Thus "words" are both determinate and indeterminate to a degree.

The depth of symbols in CWW, i.e., a degree assignment, or belonging, to a qualitative state or property, primarily covers the predicates and/or implicates of a symbol. Thus we get the following results: The subject "man" determines the predicate "animal" with a degree assignment. That is the degree to which a "man" is an "animal." But for Peirce, this means that "humanity" entails "animality" or, if you will, that a "man" is necessarily an "animal," but to a degree in CWW. Thus with Zadeh`s CWW, we are able to specify to what degree a "man" is an "animal." It is also implied that the class or collection of men is one of the "logical divisions" or subsets of the class of animals and this in turn means that "man" falls within and partially determines the breadth of "animal" with a degree assignment.

While this is true to some extend within the physical characteristics of "man" and "animal," there is an alternate use when one states that "this man is an animal" in which case one is referring to his behaviour. But then again we need to specify to which degree "this man behaves like an "animal." Thus in such uses, we are concerned with the degree to which "this man" behaves like an "animal." Hence, we need to clarify or precisiate the degree to which each man behaves like an animal.

The breadth of a symbol is whatever a symbol applies to or whatever degree it implies that symbol. Thus in the proposition in question, the subject determines the predicate's breadth and the predicate determines the subject's depth each with a specified degree. In the indeterminate case, the term "man" does not determine the predicate "sagacious." A man is not necessarily either sagacious or necessarily "non-sagacious"; "humanity" neither implies sagacity nor non-sagacity directly. The class of men could be assigned into "logically divisible" classes of the sagacious and the non-sagacious men with different degrees depending on their behaviour patterns. Thus essentially, the most important points to retain from Peirce within the context of Zadeh's CWW and fuzzy sets and logic theories are that: In the determinate case, the subject implies the predicate as part of its logical or essential depth; and this needs to be interpreted as a matter of degree. In the indeterminate case, the subject neither implies the predicate nor its negation. However, the predicate is usually a possible "further determination" of the subject with a membership assignment, and if it is added, increases the informed depth of the subject.

It should be noted that in the determinate case, the subject, set symbol, S, represents the predicate A, i.e., its individual elements, $s \in S$, with the same properties, as part of its content, i.e., the membership $\mu A(s) = a$, $a \in \{0,1\}$ for every $s \in S$ with $\mu S(s) = a$ and that this fact is affirmed to be true. That is an individual belongs to the class of "men," S, and the same individual s belongs to the class of "animal," A, and its membership in both S and A is "a=1" if it is taken in strict physical sense in the Classical theory. But as pointed out, the meaning representation of the predicate A may require that there be a degree of precisiation to be specified in fuzzy valued set interpretation. In the example given above, "this man behaves like an animal say, to degree a."

Whereas, in the indeterminate case, the set symbol, S, represents the predicate A to a certain degree such that while $\mu S(s) = 1, \mu A(s) = a \in [0,1]$ and increases our knowledge of the individual element $s \in S$ and provides us with an "informed depth," say, a=0.6, of a particular element $s \in S$. That is an individual s belongs to the class of "men," S, one hundred percent, but the same individual belongs to the class of "sagacious men" to the degree 0.6. Furthermore the "truth" verification is the affirmation of the membership degree $\mu A(s) = a$, $a \in [0,1]$ as a value between "0" and "1", i.e., $\mu V[\mu A(s) \in [0,1]]$, depending on our knowledge of the particular element's fitness to the over all class property of the set symbolized, say by A, "sagacity," in the sense of Zadeh`s "Fuzzy Set and Logic Theory and in CWW."

Peirce goes on to raise and answer some interesting general questions about the occupants (elements) of what may be called the universality of symbols.

The first equation: "Are there any entirely indeterminate general symbols; such that
$(S)(\exists P) \sim (S \text{ is } P \text{ OR } S \text{ is } \sim P)$."
The second question: "Are there any entirely determinate symbols," such that
$(S)(\exists P) \sim (S \text{ is } P \text{ OR } S \text{ is } \sim P)$."
Peirce answers both of these questions in the negative and embraces a third, mediating, position according to which every symbol is, at lease potentially, determinate in some respects, but, of course, not in the same respect. Peirce's thesis may be represented by the conjunction of
$(S)\ (\exists P)(S \text{ is } P \text{ OR } S \text{ is } \sim P)$ AND $(S)\ (\exists P) \sim (S \text{ is } P \text{ OR } S \text{ is } \sim P)$.
There are a number of reasons why Peirce rejects the notion of an absolutely indeterminate or determinate general symbol. Peirce's most forceful remark on the subject is: "[No symbol can be] …absolutely universal, since nothing could be truly asserted [about] such a symbol, i.e., everything is a matter of degree in the sense of Zadeh. However, in the sense of Peirce's requirement, it would be quite meaningless. The principle reason such a symbol would be meaningless is quite clear. Such a symbol would violate Peirce's requirement that every symbol be capable of determining an interpretant symbol, i.e., by being capable of implying something. What is being ruled

out as unintelligible here is the notion of an absolutely simple, not analyzable, indefinable, and hence inexplicable general symbol which is the stock in trade of Classical (Cartesian) Philosophy. In classical approach, we are supposed to "just understand" the meaning of such terms without being able to convey them to others. Others must "just understand" them without further explanations. Such terms are called self-defining in order to show their affinity to an intuitionist's delight: the self-evident (self-justifying!) judgments.

Peirce rejects this; in fact, his semiotic seems to be constructed so as to deprive intuitionists of their vocabulary. The positive edge of the assertion is that there are no completely indeterminate or determinate signs. That is, all symbols are, in some way or another, analyzable, definable, and explicable to others. This view can now be interpreted and expressed with the degree assignment to information granules (Zadeh,1997) as follows:

$$(S)\ (\exists P)(S \text{ is } P \text{ OR } S \text{ is } \sim P)\ (\mu P(S) \in [0,1])$$
$$\text{AND } (S)\ (\exists P) \sim (S \text{ is } P \text{ OR } S \text{ is } \sim P)\ (1 - \mu P(S))$$

Let's next consider Peirce's attack on the notion of an absolutely determinate symbol. He calls absolutely determinate symbols "logical atoms," "incapable of logical division." Because of their indivisibility, such symbols must be absolutely singular in breadth of reference. Definite descriptions and proper names would appear to be candidates for the status of "logical atoms." But, Peirce cannot allow that any symbol actually have this status without giving up the thesis that all symbols are, at least potentially, in general, that is, indeterminate, or capable of logical division into information granules of Zadeh`s CWW (1996).

PHILOSOPHICAL GROUNDING OF FUZZY THEORIES

Fuzzy theorists and practitioners, frequently find themselves confronting significant Philosophical issues in their work. Indeed, if they are not doing so, they are probably and possibly missing out a lot. While different fuzzy theories and application approaches may be founded upon different set of philosophical presuppositions, all such theories rest upon *some epistemological and ontological* assumptions, whether explicitly acknowledged or not. In this regard, the dictionary definitions of epistemology and ontology are given as follows.

Epistemology: The study or theory of the nature and grounds of knowledge, esp., with reference to its limits and validity.

Epistemology lays the ground work for the assessment of consistency and believability of a set of propositions by either a priory or evidentiary basis. Evidentiary basis could be subjective and/or objective.

Ontology: 1) A branch of metaphysics concerned with the nature and relations of being. 2) A particular theory about the nature of being or the kinds of existents.

Ontology lays the ground for the structural equivalences and the Laws of Conservation based on assumptions of existences.

A lack of appropriate treatment of the philosophical grounding creates a situation of discord, or at least a level of misunderstanding, between fuzzy theorists and practitioners on the one hand and the crisp theorists and practitioners on the other. In fact, their lack of clearly communicating their philosophical stance has caused and still causes a lack of understanding and/or rejection of fuzzy theory by those who held on to the classical view of the world. Furthermore, it does not help researchers to explain effectively to managers and decision makers, how fuzzy theory and fuzzy system models with CWW could improve their decision-making practices. Some decision-makers are still hesitant to embrace the fuzzy system models, despite the fact that significant and important applications of fuzzy theory are implemented and installed in many electro-mechanical systems, e.g., robotics, camcorders, washing machines, train break-systems, auto-transmission systems, unmanned helicopter control and navigation, etc. As well as in financial and medical decision making problems, e.g., forecasting, time series analyses, medical diagnoses, dosage degrees of a particular medication, level of financial support, etc. But it is important to note that these successful applications are different and relatively easier when compared to Classical, two valued, decision-making systems approaches.

The problem is that more often than not the underlying philosophical assumptions are overlooked or not dealt with in a sufficiently conclusive, explicit, detailed, and reflective manner. Frequently, they are left at a vague, imprecise, i.e., in an unprecisiated, and implicit level, and occasionally they are disavowed outright as in the a theoretical stance. Yet these hidden assumptions continue to exert a highly significant influence upon the ways in which the researchers and practitioner's understanding of a particular case study or a system model, will be framed, organized, or subtly structured. In particular, Type I fuzzy theory which was very successful in fuzzy control is not capable to capture "uncertainties" embedded in real life decision problems and hence require Type II fuzzy theory models.

At times, it seems as if researchers in the fuzzy disciplines have been waiting for a philosopher or someone else to come along and help them to make their philosophical unconscious more conscious, while they have played the role of a very cooperative participant in the development of the theory and/or its applications. For this reason, I call on all fuzzy theory researchers to assess and re- assess their philosophical grounding. Here, I am only providing a particular personal view. It is limited to my particular research that shows that there are at least some equivalences that break down and some laws of conservation are re-structured and some basic Belief, Plausibility and Probability formulas need to be reassessed and re-structured.

It is well known that Lotfi A. Zadeh has provided a continuous stream of novel and seminal ideas, from fuzzy sets, to fuzzy logic, to approximate reasoning to syllogistic reasoning, to Computing With Words, CWW, and Computing With Perceptions, CWP. In this regard, we are greatly indebted to Zadeh for his continuing leadership. But very few of us have taken up some of his suggestions and clearly stated our particular stance in a systematic and constructive manner. I do not mean to state that we have not made significant progress over the last forty four years or so. We have. But we still have to do a lot more.

Many researchers and practitioners have contributed to the theory and its applications in their specific area of concern. At times, some have stated their particular assumptions. But a unified view of the theory has not been stated in explicit philosophical content. Part of the problem has been methodological. Despite the many significant developments, there have been few systematic or comprehensive attempts made to look at the complex and interweaving relationships among the philosophical and scientific issues in question. In this section, we present a methodology with which one might explore the important philosophical bases of fuzzy theories in a more structured and perhaps a more rigorous manner.

In applications, philosophical positions are taken up more or less simultaneously on several different levels of a theoretical inquiry. It is thus important for us to be able to ascertain that our positions on these different levels of inquiry are consistent with one and the other. That is, we must demonstrate that our theories have some overall coherence to them. The method presented in the next section is particularly suited to provide such demonstrations at times implicit and at other times explicit.

Underlying Philosophical Bases

An overview of a systematic approach to reviewing and observing philosophical issues of fundamental importance to fuzzy theory and its practice is presented below. This method involves an analysis of the stated or implied stances taken by any given fuzzy theory on a structured series of essential philosophical questions.

A hierarchy of levels of theoretical inquiry has been developed, and proposed which include the Ontological, the General Epistemological, Domain Specific Epistemological, and the Application Levels

(Türkşen, 2004). Each level of this hierarchy poses its own fundamental philosophical questions. Each of these levels and their questions in turn provides the philosophical "grounding" of subsequent ones. A given fuzzy theory, and the many of the philosophical pre-suppositions inherent to it, may then be illustrated and classified by exploring the set of propositions adopted by it on this series of crucial questions. The results of these inquiries may then be summarized hierarchically as will be demonstrated.

I have found this systematic framework to be a useful device with which to analyze a number of fuzzy theories, and thus to increase my reflective awareness of them. To put it another way, it is an approach that may help one to explore the largely pre-reflective, unconscious, or pre-conscious philosophical dimensions of our fuzzy theories more consciously. It may more readily allow one both to asses the internal consistency or coherence of one's theories and to philosophically compare and contrast them with others as well. Here I will treat only the Ontological, the General Epistemological levels and expand on the axiomatic foundations with an investigation of re-interpreting Classical axioms "Meta-Linguistically" to expose potentially part of Type 2 developments when one interprets Classical axioms within the scope of CWW as a matter of degree.

Ontological, The General Epistemological Levels of Theoretical Inquiry

Here we examine only the four initial levels of theoretical inquiry, two in Ontological and two in General Epistemological Levels, and their questions, where the essential question thought pertinent to each level of inquiry. In this discussion an important issue is the language we use to state our theoretical inquiry. In this sense, once again Prof. Zadeh (1996-2001) has pointed out the research direction to be Computing With Words, CWW, and the need to develop Precisiated Natural Language, PNL.

The bottom levels in this theoretical inquiry are foundational to others: positions from Level 1 form the "grounding" or the conditions for the possibilities of positions on Level 2; those of 2 ground 3; etc, They are to be read from bottom level up: from 1 to 4. Thus, insights and theories are seen to rest upon a series of positions taken on each of the supporting levels 1 through 4. Let us next proceed to examine this theoretical inquiry level by level.

Ontological Level

There are two sub-levels in the Ontological Level. They are called Level 1 and Level 2. At the bottom on Level 1, theoreticians of any sort must address the most fundamental of philosophical questions: "Is there any such thing as fuzziness independent or partially independent of us?" As well, "Is there a fuzzy truth?" or, "Is there any absolute truth?" These questions form a foundation about the existence of Reality for further higher levels of inquiry. It seems obvious that whether one answers yes or no to these questions, it will have profound implications for all other levels of the theory. Type of theories and science that we propose and construct in fact depend on whether we answer "yes" or "no" to these questions. In deed, if one answers yes to these questions, it is arguable that classical theories and science has to be re-assessed and must be rendered relevant on a new grounding. As such, this level is considered most fundamental or foundational. It is well known that Classical set and logic theorist's stance is that there is the absolute Truth and that there is a crisply defined Reality that exists independent of us. Whereas the stance that fuzzy theorist's take is that there is no absolute truth and that there is a fuzzily defined Reality beginning with Zadeh's seminal paper (1965), i.e., that the Truth is a matter of degree and that the Reality is dependent on our perceptions (Zadeh, 1999). More generally, everything in our world of perception can only be assessed to a degree. At this level we have to inquire "What PNL, Precisiated Natural Language, explicates reality?" What explicates reality more precisely? Which representation of linguistic variables and their linguistic connectives are more realistic? At this junction of our history, there are essentially two alternatives, i.e., crisp or fuzzy representation of linguistic variables and their connectives that explicate reality. Possibly there are more alternatives yet to be discovered in the future to expose the hidden mysteries of "indeterminate cases" in the sense of Peirce.

Still within the realm of Ontology at Level 2, a further, higher-level question then arises: "What is our position or relation to that Reality?" Are we originally separated or apart from it, or is it the very essence of our being *relational* in this respect? Some philosophical and scientific traditions take up stances very different from others on this still quite a fundamental level. If it is relational, then "What is the nature of that relation?"

The Classical view is that our *relational* being to Reality is all or none. That is the elements of reality and their belonging to a set is "all or none." As well as the relation of these elements between sets is "all or none." The Fuzzy view is that our *relational* being to reality is a "*matter of degree.*" That is the elements of reality in their belonging to a set as well as in their relation to each other between sets is a "*matter of degree,*" i.e., there are partial memberships in a set and partial degrees in participating in relations between sets. Furthermore, the degrees of truth of these membership values are also partial. This is compatible and in agreement with the Level 1 stance that partial membership, partial participation in relations and partial truthoods are all perception based and expressed in our use of words and thus Computing With Words, CWW, that capture imprecision in set memberships and such imprecision in combination of concepts via combination of fuzzy sets are made by imprecise linguistic connectives.

Our position is that until we are able to express our knowledge with the appropriate linguistic expressions, we will not be able to capture the true nature of our relation to Reality. At this level, our inquiry is to be stated as:

"What linguistic expressions capture our positions to realty?"

"What PNL expressions capture our positions to reality?"

"What are the basic equivalences and the Laws of Conservation that capture our position to reality?"

General Epistemological Level

There are also two sub-levels here. Let us call them Level 3 and 4. Next level, Level 3, above the Ontological level is the first level of General Epistemology which asks questions about "truth and knowledge." At this level the questions of general epistemology ask, "What is our access to truth or knowledge? Where is the truth to be found in our paradigm? How or from what is it constituted?" These questions are addressed on this level in order to deal with the nature of human knowing and knowledge in general. How do we acquire knowledge: absolutely or partially? That is once we take a stance on a description of concepts and/or verification of their representation, i.e., "Truthood", being absolute or partial, and then we have to explicitly state how we obtain it. At this level, we have to ask:

"What linguistic encoding allows us to access the truth or the knowledge?"

Based upon the stances adopted on Level 3 and still within the realm of General Epistemology will be questions of General Validity at Level 4: "Given our General Epistemological position on Level 3 about "truth and knowledge," "How do we validate our knowledge?" How do we know it is true? What criteria do we use to assess its truth-value?" Again these questions are asked from the standpoint of the position and limits on human being or human knowledge in general. At this level, we have to ask:

"What linguistic expressions cause the assessment of truth and knowledge?"

As well, we have to ask:

(1) What accounts as good, strong, supportive evidence for Belief? What is the degree of Belief?

This requires that we have to come up with a "good" "Explication" of criteria of evidence or its justification in terms of a "matter of degree."

Next, we have to ask:

(2) What is the connection between a Belief being well-supported by good evidence, and the likelihood that it is true? What is the degree of its likelihood and its degree of truth?

This inquires into a new definition of "Ratification" and "Verification" criteria and their assessment to a degree. In particular, we should investigate "Belief" related assessments for "Ratification" and "Verification."

A FOUNDATION FOR COMPUTING WITH WORDS: META-LINGUISTIC AXIOMS

In this section, Meta-Linguistic axioms are proposed as a foundation for Computing With Words, CWW, (Zadeh, 1996-2001) as an extension of fuzzy sets and logic theory. Over the last 44 plus years, we have discussed and made considerable progress on the foundations of fuzzy setand logic theory and their applications in domains of mainly fuzzy control and partially fuzzy decision support systems. But in all these works, we generally have started out with the classical axioms of classical set and logic theory which are expressed in set notation and then relaxed some of these axioms, such as distributivity, absorption, idempotency, etc., in order to come up with the application of t-norms and t-conorms in various domains.

In all of this past work, we all have continued to use the classical axiomatic expressions of crisp set theory. That is, we said for example for certain t-norms and t-conorms, say distributivity, or idempotency, etc, does not hold. To say that a particular axiom "holds or does not hold" is contrary to the basic principle of fuzzy theory. If we are sincere in our basic principle which states that "all are a matter of degree" in fuzzy theory, then to say that a certain axiom "holds or does not hold"contradicts the position that "all are a matter of degree" . In fact, it speaks against our basic principle. In this regard therefore, we ought to say that all "Meta-Linguistic axioms hold as a matter of degree." For this reason, in this paper, we propose that a unique foundation for CWW can be established by re-stating the original classical axioms in terms of "Meta-Linguistic" expressions where linguistic connectives "AND","OR" are expressed linguistically as opposed to their set theoretic symbols "$\cap$", "$\cup$", respectively. These "Meta-Linguistic" expressions can then be interpreted in terms of their Fuzzy Disjunctive and Conjunctive Canonical Forms, i.e., FDCF and FCCF (Türkşen, 1986-2002).

Next we explore the consequences of this proposal when "Meta-Linguistic" expressions are interpreted with their Fuzzy Disjunctive and Conjunctive Canonical Forms, FDCF, FCCF, respectively.

In our previous writings (Türkşen, 1986-2002), we have explored various aspects of FDCF and FCCF, including their generation, their non-equivalence, i.e., FDCFi(.) $\subseteq$ FCCFi(.), i=1,…,16, for the sixteen well known linguistic expressions that form the foundation of any set and logic theory. We have also explored that, for specific cases of t-norms and t-conorms, that are strict and nilpotent Archimedean, we get: FDCFi(.) $\subseteq$ FCCFi(.) (Türkşen and Bilgiç, 1993).

In this paper, in particular, we explore in detail, the consequences of re-stating the axioms of the classical theory as "Meta-Linguistic" expressions in the development of a foundation for CWW proposed by Zadeh (1999-2001) where symbols represent "words" and connectives are linguistic "AND" , "OR".

As a result, we show that new formulas are generated in fuzzy set and logic theory as a new foundation for CWW. This demonstrates the richness and expressive power of fuzzy set and logic theories and CWW that collapse into the classical theory under restricted assumptions of reductionism. Classical theory proposes that $\mu A : X \rightarrow \{0,1\}$ in contrast to the basic principle of fuzzy set theory which proposes that $\mu A : X \rightarrow [0,1]$. That is we obtain the equivalence of the Disjunctive and Conjunctive Normal Forms, DNFi(.)= CNFi(.), i=1, … ,16, in the classical set and logic theory axioms because μ:X $\rightarrow \{0,1\}$.

In our opinion, the break down of these classical equivalences, i.e., non-equivalences, are important in establishing the foundations of fuzzy set theories and the basic formulations of Computing With

Words. This break-down and generation of additional formulas expose part of the uncertainty expressed in the combination of concepts that are generated by linguistic operators, "AND", "OR". As well, it allows us to state that the meta-linguistic axioms hold as "a matter of degree" staying true to the basic principle of fuzzy theory.

Meta-Linguistic Axioms

In order to form a sound foundation for the research to be conducted in Computing With Words, CWW, (Zadeh, 1999-2001), we believe, it is rather necessary that we begin with a statement of the basic axioms stated in form of "Meta-Linguistic" expressions. In particular, we propose that we re-state the classical axioms shown in Table 1 in terms of their "Meta-Linguistic" expressions shown in Table 2.

It is to be noted that in Table 1, in the Axioms of Classical Set and Logic Theory, A, B, C stand for classical sets such that, for example, $\mu A(x)=a\in\{0,1\}$, where $\mu A(x)=a$ is the crisp membership value of every $x\in X$, the universe of discourse X, c(.) is involutive complementation operator in the set domain which corresponds to the standard negation, n(a)=1-a, where n(.) is the involutive negation operator in the membership domain. Furthermore, "$\cap$", "$\cup$," are set theoretic "intersection," "union" operators and are taken in one-to-one correspondence to the linguistic operators "AND", "OR", respectively, in the classical reductionist perspective.

Table 1. Axioms of Classical Set and Logic Theory, Where A, B, C are Crisp, Two Valued Sets and c(.) is the Standard Complement, "$\cap$", "$\cup$" are Set Notations Which Stand for "AND", "OR" in a One-to-one Correspond, Respectively, X is the Universal Set and ϕ is the Empty Set

1 Involution:	$c(c(A)) = A$
2 Commutativity:	$A\cup B = B\cup A$,
3	$A\cap B = B\cap A$
4 Associativity:	$(A\cup B)\cup C = A\cup(B\cup C)$,
5	$(A\cap B)\cap C = A\cap(B\cap C)$
6 Distributivity:	$A\cup(B\cap C) =(A\cup B)\cap(A\cup C)$,
7	$A\cap(B\cup C) =(A\cap B)\cup(A\cap C)$
8 Idempotency:	$A\cup A = A$,
9	$A\cap A = A$
10 Absorption:	$A\cup(A\cap B) = A$,
11	$A\cap(A\cup B) = A$
12 Absorption by X and ϕ:	$A\cup X = X$,
13	$A\cap\phi = \phi$
14 Identity:	$A\cup\phi = A$,
15	$A\cap X = A$
16 Law of contradiction	$A\cap c(A) = \phi$
17 Law of excluded middle	$A\cup c(A) = X$
18 De Morgan Laws:	$c(A\cap B) = c(A)\cup c(B)$,
19	$c(A\cup B) = c(A)\cap c(B)$

Whereas in Table.2, in "Meta-Linguistic" expressions of the proposed Axioms for CWW, A, B, C stand for fuzzy sets which are linguistic terms of linguistic variables, such that, for example, $\mu A(x)=a\in[0,1]$ where $\mu A(x)=a$ is the fuzzy membership value of every $x\in X$, NOT(.) is a linguistic negation operator, which will be taken to be equivalent to the involutive negation of the classical theory, i.e., for the purposes of this paper, NOT(.)=c(.) and hence n(a)=1-a. However, the linguistic "AND", "OR" operators will be taken as linguistic connectives which do not map in a one-to-one correspondence to "$\cap$", "$\cup$", respectively, which are symbols of the classical set theory that map to t-norms and t-conorms, respectively, within the perspective of fuzzy theory.

This notion that linguistic "AND", "OR" do not correspond in a one-to-one mapping to "$\cap$", "$\cup$", respectively, is supported by Zimmermann and Zysno (1980) experiments and our investigations on "Compensatory 'AND'" (Türkşen, 1992). It should be recalled that when "Meta-Linguistic" expressions are represented in terms of FDCF and FCCF, Fuzzy Disjunctive Canonical Forms and Fuzzy Conjunctive Canonical Forms, respectively as shown in Table 3. They are no longer equivalent, i.e., FDCFi(.) $\subseteq$ FCCFi(.), i=1,...,16, for the sixteen basic natural language expressions of any two sets and logic theory (Türkşen, 1986-2002) which are shown in Table 4. This is true in particular, for example, for 3rd and 6th expressions shown in Table 4, i.e., "A OR B", and "A AND B," respectively. These two expressions and their FDCF and FCCF expressions are essential

Table 2. "Meta-Linguistic" Expression of the Axioms for CWW Where A, B, C are Fuzzy Sets and Stand For Linguistic Terms of Linguistic Variables, NOT(.) is the Complementation Operator,"AND", "OR" are Linguistic Connectives That are Not in a One-to-one Correspondence with "$\cap$", "$\cup$", Respectively

1 Involution:	NOT(NOT(A)) vs A
2 Commutativity:	A AND B vs B AND A,
3	A OR B vs B OR A
4 Associativity:	(A AND B) AND C vs A AND (B AND C),
5	(A OR B) OR C vs A OR (B OR C)
6 Distributivity:	A AND (B OR C) vs (A AND B) OR (A ANDC),
7	A OR (B AND C) vs (A OR B) AND (A OR C)
8 Idempotency:	A AND A vs A,
9	A OR A vs A
10 Absorption:	A OR (A AND B) vs A,
11	A AND (A OR B) vs A
12 Absorption by X and $\varnothing$:	A OR X vs X,
13	A AND $\varnothing$ vs $\varnothing$
14 Identity:	A OR $\varnothing$ vs A,
15	A AND X vs A
16 Law of Contradiction:	$\varnothing \subseteq$ A AND NOT(A)
17 Law of Excluded middle:	A OR NOT(A) $\subseteq$ X
18 De Morgan's Laws:	NOT(A OR B) vs NOT(A) OR NOT(B)
19	NOT(A AND B) vs NOT(A) AND NOT(B)

Table 3. Classical Disjunctive Normal and Fuzzy Disjunctive Canonical Forms, DNF and FDCF and Classic Conjunctive Normal and Fuzzy Conjunctive Canonical Forms, CNF and FCCF, Where $\cap$ is a Conjunction, $\cup$ is a Disjunction and c is a Complementation Operator in the Set Domain

Table 3.a. Fuzzy Disjunctive Canonical Forms / Disjunctive Normal Forms and Fuzzy Disjunctive Canonical Forms/Disjunctive Normal Forms

1	$(A\cap B) \cup (A\cap c(B)) \cup (c(A) \cap B) \cup (c(A) \cap c(B))$
2	$\varnothing$
3	$(A\cap B) \cup (A\cap c(B)) \cup (c(A) \cap B)$
4	$(c(A) \cap c(B))$
5	$(A\cap c(B)) \cup (c(A) \cap B) \cup (c(A) \cap c(B))$
6	$(A\cap B)$
7	$(A\cap B) \cup (c(A) \cap B) \cup (c(A) \cap c(B))$
8	$(A\cap c(B))$
9	$(A\cap B) \cup (A\cap c(B)) \cup (c(A) \cap c(B))$
10	$(c(A) \cap B)$
11	$(A\cap B) \cup (c(A) \cap c(B))$
12	$(A\cap c(B)) \cup (c(A) \cap B)$
13	$(A\cap B) \cup (A\cap c(B))$
14	$(c(A) \cap B) \cup (c(A) \cap c(B))$
15	$(A\cap B) \cup (c(A) \cap B)$
16	$(A\cap c(B)) \cup (c(A) \cap c(B))$

Table 3.b. Fuzzy Conjunctive Canonical Forms / Conjunctive Normal Forms and Fuzzy Conjunctive Canonical Forms/Conjunctive Normal Forms

1	I
2	(A∪B) ∩ (A∪c(B)) ∩ (c(A)∪B) ∩ (c(A)∪c(B))
3	(A∪B)
4	(A∪c(B)) ∩ (c(A)∪B) ∩ (c(A)∪c(B))
5	(c(A)∪c(B))
6	(A∪B) ∩ (A∪c(B)) ∩ (c(A)∪B)
7	(c(A)∪B)
8	(A∪B) ∩ (A∪c(B)) ∩ (c(A)∪c(B))
9	(A∪c(B))
10	(A∪B) ∩ (c(A)∪B) ∩ (c(A)∪c(B))
11	(A∪c(B)) ∩ (c(A)∪B)
12	(A∪B) ∩ (c(A)∪c(B))
13	(A∪B) ∩ (A∪c(B))
14	(c(A)∪B) ∩ (c(A)∪c(B))
15	(A∪B) ∩ (c(A)∪B)
16	(A∪c(B)) ∩ (c(A)∪c(B))

Table 4. Meta-Linguistic Expressions of Sixteen Possible Combinations of any Two Sets, a and B Expressed in Natural Language

1	UNIVERSE
2	EMPTY SET
3	A OR B
4	NOT A AND NOT B
5	NOT A OR NOT B
6	A AND B
7	A IMPLIES B
8	A AND NOT B
9	A OR NOT B
10	NOT A AND B
11	A IF AND ONLY IF B
12	A EXCLUSIVE OR B
13	A
14	NOT A
15	B
16	NOT B

in reinterpreting "Meta-Linguistic" axioms stated in Table 2. They can be observed in Table 3 but it should be recalled that FDCFi(.)=DNFi(.) and FCCFi(.)=CNFi, i=1,…,16 *in form only but not in content.*

Consequences of the Proposed Meta-Linguistic Axioms

In order to appreciate the consequences of the proposed Meta-Linguistic Axioms for CWW, we first very briefly review the classical axioms and the usual use of them in our investigation. After this, we state the consequences of the proposed Meta-Linguistic Axioms for CWW.

Classical Axioms

It is well known that DNFi(.)=CNFi(.), i=1,…,16, in classical theory. Thus, in Classical applications, one always use the shortest of these two forms.

For example:

(1) For "A AND B", we get:

DNF(A AND B) = A ∩ B,

CNF(A AND B) = (A ∪ B) ∩ (c(A) ∪B) ∩ (A ∪c(B)),

together with the equivalence of DNF and CNF, i.e.,

DNF(A AND B) = CNF(A AND B).

But in all our calculations, we use only "A∩B" for a representation of "A AND B" in the classical set domain. This is a conventional habit.

(2) For "A OR B", we get:

DNF(A OR B) = (A∩B) ∪ (c(A) ∩ B) ∪ (A ∩c(B)),

CNF(A OR B) = A ∪ B,

together with equivalence of DNF(A OR B) = CNF(A OR B).

But again in all our calculations, we use only "A∪B" for a representation of "A OR B" in the classical set domain. Again this is a conventional habit. This habitual use of the short hand form is applied to all the remaining linguistic combination.

Furthermore, this habit of using the short hand form of these combinations was carried out by most fuzzy researchers in their applied as well as theoretical investigations without any inquiry of what happens to the longer versions, i.e.,

DNF(A OR B) and CNF(A AND B).

However, the investigations carried out by Türkşen (1986-2002) and Türkşen, et.al.(1999) Resconi and Türkşen (2001) indicate that we ought to use both the FDCFi(.) and FCCFi(.), i=1, … ,16, because the equivalence no longer holds in fuzzy theory, i.e., FDCFi(.) $\subseteq$ FCCFi(.). For each of the proposed "Meta-Linguistic" axioms for CWW stated in Table 2, one must investigate the consequences of this nonequivalence which is an important issue known as the break-down of classical equivalences in fuzzy theory.

Investigation of Meta-Linguistic Axioms

In fuzzy theory and its applications in CWW, most researchers continue the usual habit of using the shortest form of classical axioms by directly fuzzifying all the classical axioms. That is as in the classical theory, "A AND B" is directly taken to be "A∩B" but fuzzified, "A OR B" is directly taken to be "A∪B" but fuzzified. But the other longer forms are ignored or not considered either because of habit or because most of us are generally short sighted. In addition most of us used to state that certain axioms hold and others do not hold. But such a stance is not in the spirit of fuzzy theory. As all of us believe, or ought to believe in fuzzy theory, "all are or ought to be a matter of degree". This usual habit of use continues to persist in most of the current research and applications despite the fact that the equivalences break down in fuzzy theory, e.g., FDCFi(.) $\subseteq$ FCCFi(.), i=1, … ,16, that have been published in various paper over about the last twenty plus years or so (Türkşen, 1986-2002). Thus one must take into the account the fact that these of the Fuzzy Disjunctive and Conjunctive Canonical forms are not equivalent. Therefore one has to realize that the interpretation of the proposed Meta-Linguistic Axioms must be expressed in two distinct forms in set symbolic notation and must give two distinct results in computational, numeric, domain with the application of t-norms and t-conorms. Furthermore these two distinct forms must be interpreted in the spirit of fuzz theory to state that they hold "as a matter of degree."

Next we investigate, both the FDCF and FCCF versions of the a few Meta-Linguistic Expressions as examples to demonstrate the fact that they hold in "Interval- Valeud Type 2 Fuzzy Sets"when the Axioms are interpreted linguistically in CWW paradigma. It should be noted that there are four alternative forms we must investigate since one can form FDCF vs. FDCF, FDCF vs. FCCF, FCCF vs. FCCF, FCCF vs. FDCF. For some axioms we must investigate all four forms while for some other we need investigate only two of these alternatives.

Fuzzy Involution: For the purposes of this paper, we take NOT(.) = c(.). Hence the involution axiom holds as specified, i.e., n(n(a))=a but as a matter of degree. That is there is no new interpretation of this axiom at this writing. In the future, when we investigate other linguistic negation operators, this will probably produce some new results as it should.

For our purpose in this paper, an example is this: if a =0.4 then n(a)=0.6, n(n(a))=0.4.

In order to demonstrate our perspective in this regard, we provide a few example cases from Commutativity and Associativity axioms only. But the method of investigation laid out here can be Applied to all Axioms.

Fuzzy Commutativity:

There are two Meta-Linguistic Commutativity axioms:

"A AND B = B AND A," and "A OR B = B OR A"

Now, we know that
FDCF(A AND B) $\subseteq$ FCCF(A AND B) and
FDCF(A OR B) $\subseteq$ FCCF(A OR B).

Therefore, we obtain two set theoretic axioms of the Commutativity in fuzzy theory for CWWfor these two Meta-Linguistic Axioms, (MLA).

Fuzzy Commutativity with "AND":

(a) FDCF(A AND B) vs. FDCF(B AND A), by a substitution of their fuzzy set symbols, we get:
i.e., A∩B vs. B∩A.

(b) FCCF(A AND B) vs. FCCF(B AND A), again by a substitution, we get:

i.e., (A∪B)∩(c(A)∪B)∩(c(B)∪A) vs. (B∪A)∩(c(B)∪A)∩(B∪c(A))

Fuzzy Commutativity with "OR":

(a) FDCF(A OR B) vs. FDCF(B OR A),

i.e., $(A\cap B)\cup(c(A)\cap B)\cup(A\cap c(B))$ vs. $(B\cap A)\cup(c(B)\cap A)\cup(B\cap c(A))$

(b) FCCF(A OR B) vs. FCCF(B OR A),

i.e., $A\cup B$ vs. $B\cup A$

Therefore, Fuzzy Commutativity holds fuzzily as a matter of degree in two separate forms of the MLA Commutativity axiom. This in turn exposes an uncertainty region for the Fuzzy Commutativity axioms.

Numerical Illustrations

We will illustrate the effect of interpreting meta-linguistic axioms with fuzzy theory for the following values of a=0.3, b=0.8, c=0.4 in all the numerical examples in the rest of the paper.

Illustration - Fuzzy "AND" Commutativity

For the well-known t-norm and t-conorm De Morgan Triples, we obtain:

(i) Algebraic {Sum,Product, StN}, one gets:

(a) μ [FDCF(A AND B)] = 0.24

(b) μ[FCCF(A AND B)] = 0.355

Thus there is an interval-valued Type-II result of [0.24,0.355].

(*ii*) *Lucasiewics {L1,L2 , StN}, where L1 is the sum and L2 is the product one gets:*

(a) μ[FDCF(A AND B)] = 0.1

(b) μ[FCCF(A AND B)] = 0.5

Thus there is an interval-valued Type - II result of [0.1,0.5].

Illustration - Fuzzy "OR" Commutativity

For the well-known t-norm and t-conorm De Morgan Triples we obtain:

(*i*) *{Max, Min, StN}, one gets:*

(a) μ[FDCF(A OR B)] = 0.7

(b) μ[FCCF(A OR B)] = 0.8

Thus there is an interval-valued Type-II result of [0.7,0.8].

(*ii*) *Algebraic {Sum,Product, StN}, one gets:*

(a) μ[FDCF(A OR B)] ≅0.7

(b) μ[FCCF(A OR B)] ≅0.9

Thus there is an interval-valued Type-II result of [0.7,0.9].

(*iii*) *Lucasiewics {L1,L2 , StN}*:

(a) μ[FDCF(A OR B)] = 0.6

(b) μ[FCCF(A OR B)] = 1.0

Thus there is an interval-valued Type-II result of [0.6,1.0].

Interpretation:

These interval-valued TypeII numerical results demonstrates that "Commutativity Axiom" holds in a narrow range for {*Max, Min, StN*} De Morgan Triple and it gets larger as we move toward Lucasiewicz De Morgan Triple of {*L1,L2 , StN*}.

Fuzzy Associativity

There are two ML Associativity axioms.

(1) (A AND B) AND C vs. A AND (B AND C) and

(2) (A OR B) OR C vs. A OR (B OR C)

Recall that, FDCF(A AND B) ≠ FCCF(A AND B) and

FDCF(A OR B) ≠ FCCF(A OR B)

Therefore, we obtain two set theoretic axioms of associativity in fuzzy theory for CWW for thesetwo MLA.

Fuzzy Associativity with "AND"

(a) Let us first investigate the fuzzy associativity with FDCF's:

FDCF[FDCF(A AND B) AND C] vs. FDCF[A AND FDCF (B AND C)]

We find that:

FDCF[FDCF(A AND B) AND C)] vs. FDCF[A AND FDCF(B AND C)]

i.e., we get $(A\cap B)\cap C = A\cap(B\cap C)$

This version holds to a fuzzy degree with the associativity property of t-norms and t-conorms.

(b) Let us next investigate the fuzzy associativity with FCCF's, i.e.,

FCCF[FCCF(A AND B) AND C)] vs. FCCF[A AND FCCF(B AND C)]

Recall that, we have:

FCCF(A AND B)
$= (A\cup B)\cap(c(A)\cup B)\cap (A\cup c(B))$
FCCF(B AND C) $= (B\cup C)\cap(c(B)\cup C)\cap(B\cup c(C))$

Therefore, with left hand side we get:

FCCF[FCCF(A AND B) AND C]

$=$ [FCCF(A AND B)$\cup$C] $\cap$[c[FCCF(A AND B)]$\cup$C] $\cap$[FCCF(A AND B)$\cup$c(C)]

$=\{[(A\cup B)\cap(c(A)\cup B)\cap(A\cup c(B))]\cup C\}\cap\{c[(A\cup B)\cap(c(A)\cup B)\cap(A\cup c(B)]\cup C\}\ \cap\{[(A\cup B)\cap(c(A)\cup B)\cap(A\cup c(B))]\cup c(C)\}$

With right hand side, we get:

FCCF[A AND FCCF(B AND C)]
=[A$\cup$FCCF(B AND C)]$\cap$[c(A)$\cup$FCCF(B AND C)] $\cap${A$\cup$c[FCCF(B AND C)]}
$=\{A\cup[(B\cup C)\cap(c(B)\cup C)\cap(B\cup c(C))]\}\ \cap\{c(A)\cup[(B\cup C)\cap(c(B\cup C)\cap(B\cup c(C)]\}\ \cap\{A\cup c[(B\cup C)\cap(c(B\cup C)\cap(B\cup c(C)]\}$

It should be noted and it is clear and straight forward to drive and observe that in the first case, i.e., (a) FDCF version, the interpretation of the associativity equality holds for all t-norms and t-conorms of the Proposed Meta-Linguistic Axioms to a fuzzy degree. However, it is also clear that in the second case, i.e., case (b) FCCF version, the interpretation of associativity does hold to an interval of a fuzzy degree.

Illustrations - Fuzzy Associativity with "AND" in case (a):

For the well-known t-norm De Morgan Triples we obtain:

We note that in case (a) two alternate forms hold to the same fuzzy degree because the results and set expressions, $(A \cap B) \cap C$ vs. $A \cap (B \cap C)$ hold with the t-norm , t-conorm property of associativity.

Illustration - Fuzzy Associativity with "AND" in case (b):

For the well-known t-norm De Morgan Triples we get:
This is again a singleton Type 1 result.
(i) Algebraic {Sum,Product, StN}, one gets:
μ[FCCF [FCCF(A AND B) AND C]]= 0.39..
μ[FCCF [A AND FCCF(B AND C)]] = 0.33
Thus there is an interval-valued Type-2 result of μ[0.33,0.39],

This is interval-valued TypeII result.
(ii) Lucasiewics {L1,,L2 , StN}:
μ[FCCF [FCCF(A AND B) AND C]]= 0.8
μ[FCCF [A AND FCCF(B AND C)]]= 0.6
Thus there is an interval-valued Type-2 result of μ[0.6,0.8]

Note that the resultant expressions in case (b) is not a property of the t-norms and t-conorms. Instead of the result holding to a specific fuzzy degree, they hold to fuzzy degrees in an interval that can be computed as shown above, i.e., we get Type 2 interval-valued result which indicates that there is an interval of uncertainty in which the Fuzzy Associativity with "AND" holds to a fuzzy degree.

Fuzzy Associativity with "OR"

(a) First let us investigate the fuzzy associativity for

((A OR B) OR C)=(A OR (B OR C)) with FDCF's:

FDCF[FDCF(A OR B) OR C)] vs.
FDCF[A OR FDCF(B OR C)]

Recall that, we have:

FDCF(A OR B) $= (A\cap B)\cup(c(A)\cap B)\cup(A\cap c(B))$
FDCF(B OR C) $= (B\cap C)\cup(c(B)\cap C)\cup(B\cap c(C))$

Therefore, with left hand side we get:

FDCF[FDCF(A OR B) OR C]

= [FDCF(A OR B)∩C] ∪[c[FDCF(A OR B)]∩C] ∪[FDCF(A OR B)∩c(C)]

={[(A∩B)∪(c(A)∩B)∪(A∩c(B))]∩C} ∪{c[(A∩B) ∪(c(A)∩B)∪(A∩c(B)]∩C} ∪{[(A∩B)∪(c(A)∩B)∪(A ∩c(B))]∩c(C)}

As well with right hand side, we get:

FDCF[A OR FDCF(B OR C)]

= {A∩[(B∩C)∪(c(B)∩C)∪(B∩c(C))]} ∪{c(A)∩[(B∩C)∪(c(B)∩C)∪(B∩c(C))]} ∪{A∩c[(B∪C)∪(c(B) ∩C)∪(B∩c(C))]}.

It is clear that in general, in this case, i.e., case (a), interpretation of associativity hold to fuzzydegrees in an interval.

(b) Next let us investigate the fuzzy associativity for

((A OR B) OR C) and (A OR (B OR C))with FCCF's, i.e.,:

FCCF[FCCF(A OR B) OR C)] and

FCCF[A OR FCCF(B OR C)]

Since FCCF(A OR B) = A∪B and

FCCF(B OR C) = B∪C,

we get:

(A∪B)∪C vs. A∪(B∪C),

which holds in a straight forward manner but naturally to a fuzzy degree!

Illustration - Fuzzy Associativity with "OR":

In case (a), it holds to an interval of fuzzy degrees..

(i)Algebraic {Sum,Product, StN}, one gets:

μ[FDCF [FDCF(A OR B) OR C]]= 0.651

μ[FDCF [A OR FDCF(B OR C)]]=0.600

Thus there is an interval-valued Type-2 result of μ[0.6,0.651].

(ii) Lucasiewics {L1,,L2 , StN}, one gets:

μ[FDCF [FDCF(A OR B) OR C]]= 0.2

μ[FDCF [A OR FDCF(B OR C)]]= 0.6

Thus there is an interval-valued Type-2 result of μ[0.2,0.6]

Therefore for the case (a) of the Fuzzy Associativity with "OR", we get an interval valued TypeII fuzzy degrees which demonstrate an increasing sizes of intervals of fuzzy degrees as we move away from Algebraic {Sum,Product,StN) toward Lucasiewics {L 1,L 2, StN) De Morgan Triples.

In case (b), it can be shown that it holds to a fuzzy degree.

CONCLUSIONS

In this paper, we have reviewed : 1) the perspectives of Pierce and Zadeh with regards to determinacy and indeterminacy; 2) the ontological and epistemological foundations of both the Classical and Fuzzy theories from the perspective of a theoretical inquiry. Next we have stated axiomatic positions for: 1) classical set and logic theories, 2) fuzzy set and two-valued logic theories, i.e., Type I fuzzy theory. Finally we have demonstrated a fuzzy interpretation, i.e., a Computing With Words perspective, of Meta-Linguistic Axioms to reveal part of the foundational underpinnings of Interval-Valued Type II fuzzy theory.

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